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Energy stability of the buoyancy boundary layer

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The critical value R_E of the Reynolds number R is predicted by the application of the energy theory. When $R < R_E$, the buoyancy boundary layer is the unique steady solution of the Boussinesq equations and the same boundary conditions, and is, further, stable in a slightly weaker sense than asymptotically stable in the mean. The critical value R_E is determined by numerically integrating the relevant Euler–Lagrange equations. Analytic lower bounds to R_E are obtained. Comparisons are made between R_E and R_L, the critical value of R according to linear theory, in order to demark the region of parameter space, $R_E < R < R_L$, in which subcritical instabilities are allowable.

1. Introduction

The stability of a given steady solution of the incompressible hydrodynamic equations over a region finite in at least one dimension can be examined using the method of energy (Serrin 1959). If the flow is characterized by a non-dimensional group of parameters, say, the Reynolds number R, the energy theory supplies a critical value R_E of R. It is a *sufficient condition for asymptotic stability in the mean* of the basic state that $R < R_E$. This stability is against disturbances restricted only in certain smoothness requirements and in their behaviour in directions where the geometry of the flow goes to infinity (i.e. so that the energy integrals exist). Joseph (1965, 1966) and his co-workers (Joseph & Carmi 1966; Joseph & Shir 1966; Shir & Joseph 1969) have extended these results to fluid systems involving heat transfer and subject to the Boussinesq approximation.

A *sufficient condition* that the basic state be *unstable* can be found using linear theory. When the terms quadratic in disturbance quantities are neglected compared to linear ones, one can obtain a critical value R_L of R. When $R > R_L$, the basic state is unstable.

For that region of parameter space, $R_E < R < R_L$, instability (called subcritical instability) is not ruled out, but its existence must be an effect of *finite amplitude*. When $R_L - R_E$ (which must be positive) is small, and the eigenfunctions corresponding to R_L and R_E are in some sense 'close', then one can have confidence that *both the linear and energy theories have reasonably captured the essential physics of the onset*. The ideal case is encountered in Bénard convection subject to the Boussinesq approximation. Joseph (1965) has shown in this case that the governing equations of the two theories are identical, so that $R_E = R_L$ and thus that subcritical instabilities are forbidden. When the Boussinesq equations are slightly modified, for example, by the inclusion of constant distributed heat sources (Joseph & Shir 1966), $R_E \neq R_L$, but still $R_L - R_E$ is small. Serrin (1959) also found this to be the case in the problem of the instability of Couette flow between rotating cylinders. Davis (1969*a*) has found certain *sufficient conditions* that $R_L - R_E$ be small, and their corresponding eigenfunctions be 'close' for a class of systems including those given above. These conditions depend on the symmetry properties of the time-independent linear operator of the disturbance equations and on the form of the non-linearities. Davis (1969*b*) has shown that the energy theory is applicable to surface-tension driven motions, and found that $R_L - R_E$ can again be small in certain situations.

The success, in this sense, achieved in convective, centrifugal and surface-tension driven motions, in obtaining R_E and R_L of comparable magnitude as well as their corresponding eigenfunctions 'close', has not been echoed in shear flow instability studies. For example, linear theory seems to predict stability (i.e. $R_L = \infty$) for plane Couette and Hagen–Poiseuille flows, while the energy results are for plane Couette flow $R_E = 41{\cdot}3$ (Joseph 1966) and for Hagen–Poiseuille flow $R_E = 81{\cdot}49$ (Joseph & Carmi 1969). In the latter case, the experimental value seems to be about 2100. What physical balance determines onset here?

The linearized theory can predict instability of two types. There is an inviscid, inflexional instability (Rayleigh's theorem) associated with a local maximum of the magnitude of the basic vorticity gradient (Lin 1955). If the basic flow is inviscidly stable, it can still become unstable through the existence of a critical layer (Lin 1955). The only energy theory application to flows with inflexion points is that of Carmi (1969), who considered classes of pseudo-parallel channel flows. All other shear flow applications have been on flows which are either inviscidly stable as is plane Poiseuille flow, or else on flows for which the Rayleigh criterion fails altogether. In these cases, there seems little relation between the values of R_L and R_E. This is perhaps due to the fact that the critical layer mechanism of importance in the linear theory is absent in the Euler–Lagrange equations that emerge from the energy theory.

One of the purposes of the present analysis is the application of the theory to a basic flow having points of inflexion. This flow is, according to linear theory, inviscidly unstable, and it is the hope that, if in this case the values of R_L and R_E are comparable, the physical utility of the energy theory in shear flow problems would be more fully appreciated. This hope is indeed borne out (see also Carmi 1969). In the course of this application, several modifications in the usual procedures are necessitated, since the buoyancy boundary layer flow domain has no finite dimension. As a result, several previously used estimates now fail and the definition of stability when $R < R_E$ must be slightly weakened. Furthermore, the universal stability criterion (Serrin 1959) fails. A new set of lower bounds for R_E is found.

The physical problem treated here, the buoyancy boundary layer, represents an exact solution to the Boussinesq equations of motion. Prandtl's (1952) 'mountain and valley winds in stratified air' contains this solution as a special case when the surface under consideration is vertical rather than slanted. The specific solution used here was given in Gill (1966) in connection with the problem of thermal convection in a rectangular cavity (see also Batchelor 1954). It is an approximation to the solution at the vertical midpoint of the cavity, far away from the horizontal walls at the top and bottom of the cavity. Solutions of similar form have been found by Barcilon & Pedlosky (1967) for vertical boundary layers of rotating, strongly stratified systems. The boundary layer thickness was found to be of order $E^{\frac{1}{2}}(PS)^{-\frac{1}{4}}$ where the Ekman number $E = \nu/\Omega L^2$, ν is the kinematic viscosity, Ω is the vertical component of rotation, L is a length scale, P is the Prandtl number and S measures the vertical stratification. The buoyancy boundary layer consistently recurs in stratified fluid systems sustaining horizontal temperature gradients.

A linear stability analysis has been carried out by Gill & Davey (1969). They considered only two-dimensional disturbances (although, as they note, Squire's theorem is not applicable) and computed R_L numerically as a function of the Prandtl number. They thus obtained a sufficient condition, $R > R_L$, that the buoyancy layer be unstable; but, since three-dimensional disturbances cannot be excluded, their values of R_L may be conservatively high. Gill & Davey (1969) have also shown that, while for large Prandtl numbers the instability is buoyancy driven, for small Prandtl numbers the flow is hydrodynamically unstable due to the points of inflexion in the basic flow field. The previously mentioned test for the energy method can then be made.

The values R_E given by the energy method are obtained numerically from integration of the Euler–Lagrange equations. These are compared to the linear theory of Gill & Davey (1969), and also to the experimental work of Elder (1965) on the instability of the vertical boundary layer in the convective flow in a rectangular cavity.

2. The basic flow and temperature fields

The basic state is an exact solution of the Boussinesq equations. The system represented by these fields consists of a fluid of mean density ρ_0 (when at a temperature T_0) occupying the space $x > 0$ bounded by an infinite vertical wall located at $x = 0$. The acceleration due to gravity (acting vertically downward, in the negative z-direction) $\mathbf{g} = (0, 0, -g)$, the coefficients of kinematic viscosity ν, thermal diffusivity κ and volume thermal expansion α are all assumed constant. The basic state is a uni-directional boundary layer flow against a stable linear vertical stratification. The basic stratification is established by requiring the temperatures of the solid boundary at $x = 0$ and the fluid at $x \to \infty$ to vary linearly with z, Gz where $G > 0$. The motion is driven by a horizontal temperature difference ΔT between the wall and infinity at each height z.

The Boussinesq equations (including a linear equation of state) governing the problem are the following:

$$\left.\begin{aligned} &\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}.\nabla\mathbf{V} = -\nabla(p/\rho_0) + [1-\alpha(T-T_0)]\,\mathbf{g} + \nu\nabla^2\mathbf{V}, \\ &\frac{\partial T}{\partial t} + \mathbf{V}.\nabla T = \kappa\nabla^2 T, \quad \nabla.\mathbf{V} = 0. \end{aligned}\right\} \tag{2.1 a–c}$$

The boundary conditions are given by

$$\left.\begin{aligned}\mathbf{V} = 0, \quad T &= \Delta T + Gz \quad \text{on} \quad x = 0,\\ \mathbf{V} \to 0, \quad T &\to Gz \qquad\quad \text{as} \quad x \to \infty.\end{aligned}\right\} \tag{2.1 d, e}$$

The relevant solution of system (2.1) is called the buoyancy boundary layer, and was given by Gill (1966) as follows:

$$\left.\begin{aligned}U &= V \equiv 0,\\ W &= V_0 e^{-x/L} \sin(x/L),\\ T &= (\Delta T)\, e^{-x/L} \cos(x/L) + Gz,\end{aligned}\right\} \tag{2.2 a–c}$$

where the velocity vector $\mathbf{V} = (U, V, W)$, the pressure is denoted by p,

$$L = (4\nu\kappa/\alpha g G)^{\frac{1}{4}} \quad \text{and} \quad V_0 = (\alpha g \kappa/\nu G)^{\frac{1}{2}}\,\Delta T.$$

3. The energy and entropy identities

We shall now develop the energy and entropy identities for disturbances to the basic state (2.2). Following Joseph (1965), let $(\mathbf{V}, T, p)$ be the solution of the basic state, and let $(\mathbf{V}^*, T^*, p^*)$ be any other solution to the Boussinesq equations and boundary conditions (2.1). Now let

$$\left.\begin{aligned}\mathbf{u} &= (u, v, w) = \mathbf{V}^* - \mathbf{V},\\ \phi &= T^* - T,\\ \pi &= p^* - p.\end{aligned}\right\} \tag{3.1 a–c}$$

$(\mathbf{u}, \phi, \pi)$ represents the difference between the disturbed and undisturbed states. Since both $(\mathbf{V}, T, p)$ and $(\mathbf{V}^*, T^*, p^*)$ satisfy system (2.1), we can obtain the system governing the disturbances:

$$\left.\begin{aligned}&\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\,.\,\nabla\mathbf{u} + \mathbf{V}\,.\,\nabla\mathbf{u} + \mathbf{u}\,.\,\nabla\mathbf{V} = -\nabla(\pi/\rho_0) - \alpha \mathbf{g}\phi + \nu\nabla^2\mathbf{u},\\ &\frac{\partial \phi}{\partial t} + \mathbf{u}\,.\,\nabla\phi + \mathbf{V}\,.\,\nabla\phi + \mathbf{u}\,.\,\nabla T = \kappa\nabla^2\phi,\\ &\qquad\qquad \nabla\,.\,\mathbf{u} = 0,\\ &\qquad \mathbf{u} = \phi = 0 \quad \text{on} \quad x = 0,\\ &\qquad \mathbf{u}, \phi \to 0 \quad \text{as} \quad x \to \infty.\end{aligned}\right\} \tag{3.2 a–e}$$

We shall henceforth confine our attention to disturbance functions u, v, w, ϕ, π which belong to the class $\mathscr{F}$:

$$\mathscr{F} = \left\{ f \,\middle|\, \int_{x=0}^{\infty} |f|\, dx < \infty, \right.$$

f is either

(i) periodic in y and z,

(ii) Fourier transformable in y and z,

(iii) periodic in either y or z and Fourier transformable in the other.$\Big\}$ (3.3)

For functions in $\mathscr{F}$, we define the integral over a volume $\mathscr{V}$ by

$$\int_{\mathscr{V}} f \equiv \int_{z=0}^{2\pi/l}\int_{y=0}^{2\pi/k}\int_{x=0}^{\infty} f\,dx\,dy\,dz$$

in case (i) where k and l are the wave-numbers in the y and z directions respectively. For case (ii),

$$\int_{\mathscr{V}} f \equiv \int_{z=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{x=0}^{\infty} f\,dx\,dy\,dz.$$

For case (iii), the integral will extend over a wavelength in the co-ordinate in which the functions are periodic, from $-\infty$ to ∞ in the co-ordinate in which they are Fourier transformable and from 0 to ∞ in x.

We define $\int_{\partial\mathscr{V}} f$ as the integral of f over the surface of the rectangular parallelepiped whose appropriate dimensions (depending upon whether we have cases (i), (ii) or (iii)) extend to infinity.

We now form the scalar product of (3.2a) with $\mathbf{u}$, multiply (3.2b) by ϕ and integrate over $\mathscr{V}$. We employ Green's theorem, (3.2c) and the fact that all the integrals over $\partial\mathscr{V}$ vanish and obtain the identities:

$$\left.\begin{aligned}\frac{dK}{dt} &= -\int_{\mathscr{V}}[\mathbf{u}.\mathbf{D}.\mathbf{u}+\alpha\mathbf{g}.\mathbf{u}\phi+\nu\nabla\mathbf{u}:\nabla\mathbf{u}],\\ \frac{d\Theta}{dt} &= -\int_{\mathscr{V}}[\phi\mathbf{u}.\nabla T+\kappa\nabla\phi.\nabla\phi],\end{aligned}\right\} \qquad (3.4a,b)$$

where

$$K = \int_{\mathscr{V}}\tfrac{1}{2}\mathbf{u}.\mathbf{u}, \quad \Theta = \int_{\mathscr{V}}\tfrac{1}{2}\phi^2, \qquad (3.4c,d)$$

$$\mathbf{D} = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & W_x \\ 0 & 0 & 0 \\ W_x & 0 & 0 \end{bmatrix} \qquad (3.4e)$$

and subscripts x denote differentiation. These identities are precisely those of Joseph (1965), except that now the volume $\mathscr{V}$ extends to infinity in the x-direction.

We non-dimensionalize these identities using the following scales: length $\sim L$, velocity $\sim V_0$, temperature $\sim \Delta T$, time $\sim L^2/\nu$ where L, V_0 and ΔT are defined below (2.2). We obtain

$$\left.\begin{aligned}\frac{dK}{dt} &= -\int_{\mathscr{V}}[R\mathbf{u}.\mathbf{D}.\mathbf{u}+2\mathbf{f}.\mathbf{u}\phi+\nabla\mathbf{u}:\nabla\mathbf{u}],\\ P\frac{d\Theta}{dt} &= -\int_{\mathscr{V}}[PR\phi\mathbf{u}.\nabla T+\nabla\phi.\nabla\phi],\end{aligned}\right\} \qquad (3.5a,b)$$

where $\mathbf{f} = (0,0,-1)$, the Reynolds number R is defined by

$$R = V_0L/\nu = \Delta T(4\alpha g\kappa^3/\nu^5G^3)^{\frac{1}{4}}$$

and the Prandtl number P is $P = \nu/\kappa$. The scaled basic state is given by

$$\left.\begin{aligned}W &= e^{-x}\sin x,\\ T &= e^{-x}\cos x+\frac{2z}{PR};\end{aligned}\right\} \qquad (3.5c,d)$$

and the strain rate tensor is

$$\mathbf{D} = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & W_x \\ 0 & 0 & 0 \\ W_x & 0 & 0 \end{bmatrix}. \tag{3.5e}$$

Since we will be using only non-dimensional variables henceforth, we have not distinguished between dimensional and non-dimensional quantities.

4. The maximum problem

One would usually now introduce (Joseph 1965) the coupling parameter λ and define the 'energy' functional

$$E_\lambda = K + \lambda P\Theta \quad (\lambda > 0).$$

The requirement $\lambda > 0$ ensures that K and Θ will approach zero if and only if E_λ does. One could then establish a variational principle in which the eigenvalue R_λ gives the stability criterion for a given λ. That is, if $R < R_\lambda$, stability, in the sense to be defined, is guaranteed for that λ. One could then *postulate* that all dependent variables are analytic functions of λ, use parametric differentiation, and obtain the optimal value λ^* of λ. This optimal value gives

$$R_{\lambda^*} = \sup_\lambda R_\lambda.$$

One would then use $\lambda = \lambda^*$, and have the optimal stability boundary $R_E = R_{\lambda^*}$. However, there are certain problems for which the assumption of analyticity is unfounded. The simplest case is of Bénard convection heated from above (stably stratified; Shir & Joseph 1968). In this case, there is a singularity in the dependent variables at $\lambda = 1$. This is manifested in the prediction $\lambda^* = -1$ in violation of the restriction $\lambda > 0$. Shir & Joseph (1968) also find this to be the case in thermohaline convection when the thermal field is destabilizing, but the concentration field is stably stratified. When this procedure was applied to the present problem, we were able to obtain *formally* an expression for λ^*. When $P \to 0$, we found that $\lambda^* \to -1$, while for other values of P it seemed as though λ^* would again be negative. Since it *seems* as though there is again a singularity characteristic of situations having a stably stratified diffusive field, we shall henceforth restrict our attention to the case $\lambda = 1$. At worst, this restriction will yield a stability condition which is correct but too conservative, while at best we will obtain the optimal stability boundary. The reader interested in the derivation of λ^* is referred to Dudis (1970).

We can obtain an expression for the rate of change of E_1 by adding (3.5a, b).

$$\frac{dE_1}{dt} = -D_1 + RI_1, \tag{4.1a}$$

where
$$D_1 = \int_{\mathscr{V}} [\nabla\mathbf{u} : \nabla\mathbf{u} + \nabla\phi . \nabla\phi], \tag{4.1b}$$

and
$$I_1 = -\int_{\mathscr{V}} [\mathbf{u} . \mathbf{D} . \mathbf{u} + P\phi u T_x]. \tag{4.1c}$$

From (4.1 *a*), we can write

$$\frac{dE_1}{dt} = -\left(1 - R\frac{I_1}{D_1}\right) D_1. \tag{4.2}$$

It follows from (4.2) that

$$\frac{dE_1}{dt} \leqslant -\left(1 - \frac{R}{R_E}\right) D_1 \tag{4.3}$$

whenever the maximum problem (M) is satisfied:

$$R_E^{-1} = \max_{\mathscr{S}} I_1, \tag{M}$$

$$D_1 = 1,$$

$$\mathscr{S} = \{\mathbf{u}, \phi | \mathbf{u}, \phi \text{ have continuous second partial derivatives, } \nabla . \mathbf{u} = 0,$$
$$\mathbf{u} = \phi = 0 \quad \text{on} \quad x = 0 \quad \text{and} \quad u, v, w, \phi \in \mathscr{F}\}.$$

We now reach a point of departure from previous analyses. The proof that the basic motion is asymptotically stable in the mean when $R < R_E$ does not now follow in the same way since previous proofs were dependent on the existence of *positive* numbers a_1^2 and a_2^2 in the following inequalities:

$$\int_{\mathscr{V}} \nabla\mathbf{u} : \nabla\mathbf{u} \geqslant a_1^2 \int_{\mathscr{V}} \mathbf{u} . \mathbf{u} \tag{I}$$

and

$$\int_{\mathscr{V}} \nabla\phi . \nabla\phi \geqslant a_2^2 \int_{\mathscr{V}} \phi^2.$$

Since we are now dealing with a *domain infinite in all dimensions*, these inequalities fail. In fact, we do not have stability in the same sense as before.

If $R < R_E$ in the time interval $[0, \tau]$, then if follows from (4.3) that

$$E_1(\tau) - E_1(0) \leqslant -\left(1 - \frac{R}{R_E}\right) \int_0^\tau D_1 dt. \tag{4.4}$$

If $E_1(0)$ is bounded, then (4.4) implies that $E_1(\tau)$ is uniformly bounded in τ. It then follows that $D_1 \to 0$ as $\tau \to \infty$, in the sense that

$$\lim_{\tau \to \infty} \int_0^\tau D_1 dt < \infty. \tag{4.5}$$

Because the fluid is incompressible, this implies that the disturbance vorticity approaches zero as $\tau \to \infty$ in the above sense. A further implication follows:

Even though inequalities (I) fail to hold, since $\mathscr{V}$ has no finite dimension, the following weaker inequalities *are* valid: Let $\tilde{\mathscr{V}}$ be a rectangular parallelepiped bounded in the x-direction with the wall $x = 0$ as one of its boundaries. The extent in the y- and z-directions is either a wavelength or the whole real line depending on which case of class $\mathscr{F}$ is being considered. Then, there exists positive numbers $\tilde{a}_1^2$ and $\tilde{a}_2^2$ such that

$$\int_{\tilde{\mathscr{V}}} \nabla\mathbf{u} : \nabla\mathbf{u} \geqslant \tilde{a}_1^2 \int_{\tilde{\mathscr{V}}} \mathbf{u} . \mathbf{u}, \tag{II}$$

and

$$\int_{\tilde{\mathscr{V}}} \nabla\phi . \nabla\phi \geqslant \tilde{a}_2^2 \int_{\tilde{\mathscr{V}}} \phi^2.$$

Since $\tilde{\mathscr{V}} \subset \mathscr{V}$, we have that

$$\left.\begin{aligned}\int_{\mathscr{V}} \nabla\mathbf{u} : \nabla\mathbf{u} &\geqslant \int_{\tilde{\mathscr{V}}} \nabla\mathbf{u} : \nabla\mathbf{u}, \\ \int_{\mathscr{V}} \nabla\phi . \nabla\phi &\geqslant \int_{\tilde{\mathscr{V}}} \nabla\phi . \nabla\phi.\end{aligned}\right\} \tag{4.6a, b}$$

If we combine (II) with (4.6), we obtain that

$$D_1 \geqslant \tilde{a}^2 \tilde{E}_1, \tag{4.7a}$$

where
$$\tilde{a}^2 = \min(\tilde{a}_1^2, \tilde{a}_2^2 P^{-1}) \quad \text{and} \quad \tilde{E}_1 = \int_{\tilde{\mathscr{V}}} [\mathbf{u}.\mathbf{u} + P\phi^2]. \tag{4.7b}$$

From (4.5) we have that $\tilde{E}_1 \to 0$, in the sense that

$$\lim_{\tau\to\infty} \int_0^\tau \tilde{E}_1\, dt < \infty. \tag{4.8}$$

We sum up: For any fixed value of P, if $R < R_E$, then we have stability of the basic state defined by $D_1 \to 0$ as $\tau \to \infty$ in the sense of (4.5). If $\tilde{\mathscr{V}}$ is a rectangular parallelepiped defined as by (II), then the definition of stability further means that $\tilde{E}_1 \to 0$ as $\tau \to \infty$ in the sense of (4.8). Furthermore, if $R < R_E$, then the buoyancy boundary layer represented by (2.2) is the unique steady solution of the Boussinesq equations and boundary conditions (2.1 d, e) over the class of functions $\mathscr{F}$.

It is worth mentioning that the weakened stability criterion could be strengthened to asymptotic stability in the mean, if we were to insert a second 'distant' boundary parallel to the first. The basic state would then be altered as would the numerical value of R_E.

5. The Euler–Lagrange equations

The maximum problem (M) is equivalent to the following variational equation:

$$\delta\left\{\int_{\mathscr{V}} [\mathbf{u}.\mathbf{D}.\mathbf{u} + P\phi u T_x] - \frac{2}{R}\int_{\mathscr{V}} p\nabla.\mathbf{u} + \frac{1}{R}\int_{\mathscr{V}} [\nabla\mathbf{u} : \nabla\mathbf{u} + \nabla\phi . \nabla\phi]\right\} = 0, \tag{5.1}$$

where R and $p(x, y, z)$ are Lagrange multipliers resulting from the normalization $D_1 = 1$ and the condition $\nabla.\mathbf{u} = 0$. The variations are taken over the extension of $\mathscr{S}$ formed by relaxing the condition $\nabla.\mathbf{u} = 0$. The consequent Euler–Lagrange system is as follows:

$$\left.\begin{aligned}R\mathbf{u}.\mathbf{D} + \tfrac{1}{2}RP\phi T_x \mathbf{i} &= -\nabla p + \nabla^2\mathbf{u}, \\ \tfrac{1}{2}RPT_x u &= \nabla^2\phi, \\ \nabla.\mathbf{u} &= 0,\end{aligned}\right\} \tag{5.2a–c}$$

with the boundary conditions

$$\left.\begin{aligned}\mathbf{u} = \phi &= 0 \quad \text{on} \quad x = 0, \\ \mathbf{u}, \phi &\to 0 \quad \text{as} \quad x \to \infty,\end{aligned}\right\} \tag{5.2d, e}$$

and $\mathbf{i} = (1, 0, 0)$. The smallest positive eigenvalue of R of system (5.2) gives the sufficient condition for stability which we seek. For if we let $(\mathbf{u}, \phi, p)$ be a solution of (5.2), take the inner product of $\mathbf{u}$ with (5.2*a*), the product of ϕ with (5.2*b*) and add, the result is

$$\int_{\mathscr{V}} [\mathbf{u}.\mathbf{D}.\mathbf{u} + Pu\phi T_x] = -\frac{1}{R} D_1.$$

If we use the normalization condition $D_1 = 1$ and the definitions of D_1 and I_1, we have that

$$1/R = I_1. \tag{5.3}$$

The problem of determining the stability boundary (for $\lambda = 1$) for the buoyancy boundary layer reduces to finding

$$R_E = \min \operatorname{pos} R;$$

and we have stability and uniqueness in the above sense if $R < R_E$.

6. An *a priori* bound by parametric differentiation

Let us assume that the eigenvalue $R_E(P)$ and $(\mathbf{u}(x, y, z; P), \phi(x, y, z; P))$, which solve the maximum problem (M) and the equivalent Euler–Lagrange system (5.2), are continuously differentiable functions of the Prandtl number P. Then, we have that

$$\frac{dR_E}{dP} = -\frac{2R_E}{P} \int_{\mathscr{V}} \nabla\phi . \nabla\phi. \tag{6.1}$$

Proof

Let $(\mathbf{u}, \phi)$ be a solution corresponding to R_E for any fixed value of P. Consider two such solutions, and label them with subscripts. From (5.2), we can find:

$$(a)\quad R_{E1}\int_{\mathscr{V}} \mathbf{u}_1.\mathbf{D}.\mathbf{u}_2 + \tfrac{1}{2}R_{E1}P_1\int_{\mathscr{V}} T_x\phi_1 u_2 = -\int_{\mathscr{V}} \nabla\mathbf{u}_1 : \nabla\mathbf{u}_2,$$

$$(b)\quad R_{E2}\int_{\mathscr{V}} \mathbf{u}_2.\mathbf{D}.\mathbf{u}_1 + \tfrac{1}{2}R_{E2}P_2\int_{\mathscr{V}} T_x\phi_2 u_1 = -\int_{\mathscr{V}} \nabla\mathbf{u}_2 : \nabla\mathbf{u}_1,$$

$$(c)\quad \tfrac{1}{2}R_{E1}P_1\int_{\mathscr{V}} T_x u_1\phi_2 = -\int_{\mathscr{V}} \nabla\phi_1 . \nabla\phi_2,$$

$$(d)\quad \tfrac{1}{2}R_{E2}P_2\int_{\mathscr{V}} T_x u_2\phi_1 = -\int_{\mathscr{V}} \nabla\phi_2 . \nabla\phi_1.$$

If we form the sum $(a)-(b)+(c)-(d)$, and allow the solutions to coalesce, we obtain

$$dR_E\int_{\mathscr{V}} \mathbf{u}.\mathbf{D}.\mathbf{u} + d(R_E P)\int_{\mathscr{V}} T_x u\phi.$$

We solve for dR_E/dP and find

$$\frac{dR_E}{dP} = -R_E\int_{\mathscr{V}} T_x u\phi \Big/ \int_{\mathscr{V}} [\mathbf{u}.\mathbf{D}.\mathbf{u} + PT_x u\phi].$$

If we use relations (4.1c) and (5.3), we obtain

$$\frac{dR_E}{dP} = R_E^2 \int_{\mathscr{V}} T_x u \phi.$$

The result (6.1) follows from the equation obtained by multiplying (5.2b) by ϕ and integrating over $\mathscr{V}$.

We can obtain *a priori* bounds on R_E as a function of P through (6.1). First, since $\int_{\mathscr{V}} \nabla\phi . \nabla\phi$ is positive definite, it follows that

$$\frac{dR_E}{dP} \leqslant 0. \tag{6.2}$$

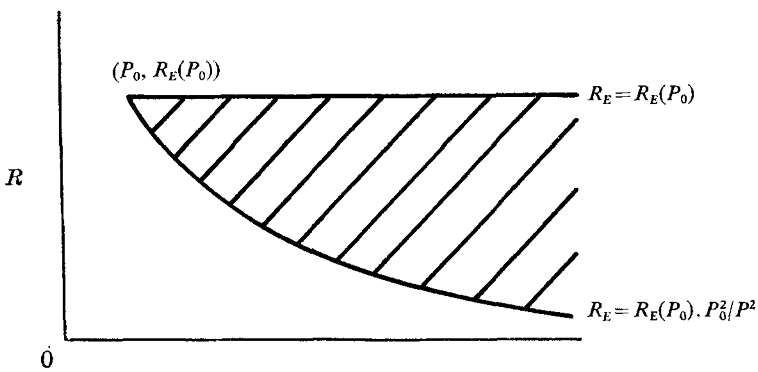

FIGURE 1. *A priori* bounds to $R_E(P)$. R_E lies in the shaded region.

Secondly, from $D_1 = 1$ we have that $\int_{\mathscr{V}} \nabla\phi . \nabla\phi \leqslant 1$. It follows that

$$\frac{dR_E}{dP} \geqslant -\frac{2R_E}{P}. \tag{6.3}$$

If we integrate (6.3) from some initial value $(P_0, R_E(P_0))$ to $(P, R_E(P))$, we have

$$R_E(P) \geqslant \frac{R_E(P_0) P_0^2}{P^2}. \tag{6.4}$$

A priori bounds (6.2) and (6.4) are illustrated in figure 1; the exact $R_E(P)$ lies in the shaded region.

7. Improved lower bounds for the stability boundary

In our discussion of the definition of stability, we mentioned that a slightly weaker definition was required when dealing with a flow domain no dimension of which is finite. The mathematical reason is that the inequalities (I) (see below (M)) required for the stronger condition fail to hold in this domain. This same failure has another consequence. A universal stability criterion (Serrin 1959) is no longer obtainable. As a result, we shall now seek another method of obtaining stronger lower bounds on $R_E(P)$ than established in § 6. To this end, we rewrite the maximum principle (M):

$$R_E^{-1}(P) = \max_{\mathscr{S}} I_1,$$
$$D_1 = 1.$$

One means of obtaining a lower bound on R_E is to maximize I_1, subject to $D_1 = 1$, over a space larger than $\mathscr{S}$. One way of obtaining such an extension is to relax the restriction $\nabla . \mathbf{u} = 0$. We can define $\mathscr{S}_1$,

$$\mathscr{S}_1 = \{\mathbf{u}, \phi | \mathbf{u}, \phi \text{ have continuous second partial derivatives, } \mathbf{u} = \phi = 0 \text{ on } x = 0 \text{ and } u, v, w, \phi, \in \mathscr{F}\},$$

and we have that

$$\mathscr{S}_1 \supset \mathscr{S}. \tag{7.1}$$

As a result of (7.1), we can define R_w by

$$R_w^{-1} = \sup_{\mathscr{S}_1} I_1, \qquad D_1 = 1, \tag{M1}$$

and we have the relationships that

$$R_w \leqslant R_E, \tag{7.2}$$

and $R_w = \inf \operatorname{pos} \bar{R}$. This extension gives rise to an equivalent set of Euler–Lagrange equations as follows:

$$\left.\begin{aligned} \tfrac{1}{2}\bar{R}W_x w + \tfrac{1}{2}\bar{R}PT_x\phi &= \nabla^2 u, \\ 0 &= \nabla^2 v, \\ \tfrac{1}{2}\bar{R}W_x u &= \nabla^2 w, \\ \tfrac{1}{2}\bar{R}PT_x u &= \nabla^2 \phi, \end{aligned}\right\} \tag{7.3a–d}$$

with the boundary conditions

$$\left.\begin{aligned} &\mathbf{u} = \phi = 0 \quad \text{on} \quad x = 0, \\ &\mathbf{u}, \phi \to 0 \qquad \text{as} \quad x \to \infty. \end{aligned}\right\} \tag{7.3e,f}$$

The mean value theorem for harmonic functions gives that $v \equiv 0$.

A still lower bound can be obtained *analytically*. Consider

$$I_1 = -\int_{\mathscr{V}} [W_x uw + PT_x u\phi].$$

I_1 can be conveniently rewritten in the form,

$$I_1 = -\int_{\mathscr{V}} \boldsymbol{\zeta} . \mathbf{B} . \boldsymbol{\zeta},$$

where

$$\boldsymbol{\zeta} = (u, v, w, \phi),$$

and

$$\mathbf{B} = \tfrac{1}{2}\begin{bmatrix} 0 & 0 & W_x & PT_x \\ 0 & 0 & 0 & 0 \\ W_x & 0 & 0 & 0 \\ PT_x & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $\mathbf{B}$ are 0 and $\pm\frac{1}{2}(W_x^2+P^2T_x^2)^{\frac{1}{2}}$. We thus have that

$$I_1 \leqslant \frac{1}{2}\int_{\mathscr{V}} (W_x^2+P^2T_x^2)^{\frac{1}{2}}\boldsymbol{\zeta}.\boldsymbol{\zeta}.$$

Since $W_x \leqslant \sqrt{2}e^{-x}$ and $T_x \leqslant \sqrt{2}e^{-x}$, we have that

$$I_1 \leqslant \frac{\sqrt{2}}{2}(1+P^2)^{\frac{1}{2}}\int_{\mathscr{V}} e^{-x}\boldsymbol{\zeta}.\boldsymbol{\zeta} \equiv J_1.$$

Define R_{ww} by $$R_{ww}^{-1} = \sup_{\mathscr{S}_1} J_1, \quad D_1 = 1. \tag{M 2}$$

We now have that $$R_{ww} \leqslant R_w \leqslant R_E. \tag{7.4}$$

The Euler–Lagrange equations for (M 2) reduce to a single scalar equation for, say, u:

$$\nabla^2 u + \tfrac{1}{2}\bar{R}[2(1+P^2)]^{\frac{1}{2}}e^{-x}u,$$
$$u = 0 \quad \text{on} \quad x = 0, \infty,$$

where $$R_{ww} = \inf \bar{R}.$$

For the periodic case in $\mathscr{F}$, we can write

$$u = \operatorname{Re}\{\hat{u}(x)\exp[i(ky+lz)]\}.$$

When u is Fourier transformable, we can write

$$u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{u}(x)\exp[i(ky+lz)]\,dk\,dl.$$

In either case (or in the mixed case (c)), we arrive at the same equation for $\hat{u}$, viz.

$$\left.\begin{aligned}\left(\frac{d^2}{dx^2} - m^2 + \tfrac{1}{2}\bar{R}[2(1+P^2)]^{\frac{1}{2}}e^{-x}\right)\hat{u} &= 0,\\ \hat{u}(0) = \hat{u}(\infty) &= 0,\end{aligned}\right\} \tag{7.5a, b}$$

where $m = (k^2+l^2)^{\frac{1}{2}}$.

This system can be transformed into the Bessel equation by letting

$$r = e^{-\frac{1}{2}x},$$

which maps the x interval $[0, \infty]$ into the r interval $[1, 0]$. The eigenfunction is $J_{2m}(\beta r)$ subject to the eigenvalue relationship,

$$\beta = \{2\bar{R}[2(1+P^2)]^{\frac{1}{2}}\} \tag{7.6}$$

for all values of the overall wave number, m, $m \neq 0$ as long as β is a zero of J_{2m}. Equation (7.6) thus determines all the eigenvalues of $\bar{R}$ as a function of m. Since $R_{ww} = \inf\bar{R}$, we need to use the lowest zero β of J_{2m}. The first zero of J_{2m} is a continuous function of m and is a monotone decreasing function of m, the limiting value approaching 2·405 (Watson 1944) as $m \to 0$. The minimum is not attained, however, since J_0 does not satisfy the boundary condition as $r \to 0$; only $\lim_{m\to 0} J_{2m}(\beta r)$ need be found. Thus, we have that

$$[2R_{ww}[2(1+P^2)]^{\frac{1}{2}}]^{\frac{1}{2}} = 2{\cdot}405,$$

and hence that $$R_{ww} = 2{\cdot}04(1+P^2)^{-\frac{1}{2}}. \tag{7.7}$$

The relation (7.7) gives a lower bound to R_E. As a test of the numerical scheme to be used later (Runge–Kutta–Gill, see appendix A), the system (7.5) was numerically integrated. Furthermore, the better lower bound R_w governed by system (7.3) was computed numerically and it was found that the ratio R_w/R_{ww}

	Numerical		Analytical	
P	R_w	PR_w	R_{ww}	PR_{ww}
0·0	4·05	0·00	2·04	0·00
0·1	4·03	0·403	2·03	0·203
1·0	3·09	3·09	1·45	1·45
10·0	0·418	4·18	0·203	2·03
100·0	0·0420	4·20	0·0204	2·04
1000·0	0·00420	4·20	0·00204	2·04

TABLE 1. Lower bounds R_w and R_{ww} for various values of P.

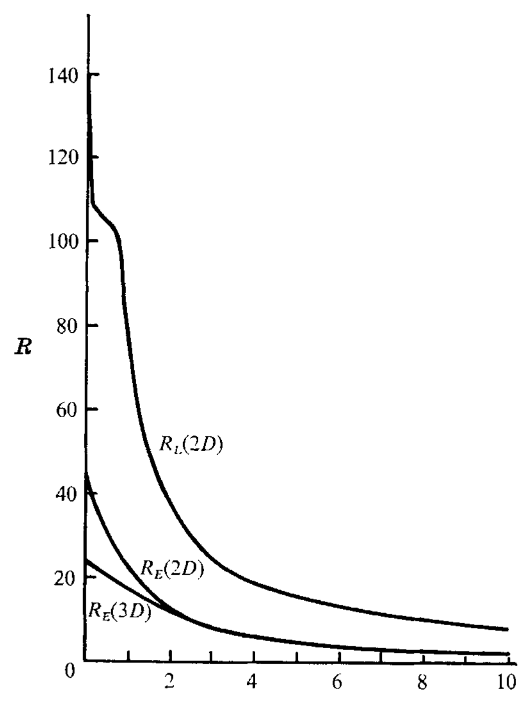

FIGURE 2. The stability boundaries; $R_E(2D)$, $R_E(3D)$ and $R_L(2D)$ as functions of P.

was about two. As is evident from (7.7), and as was also found for R_w numerically for large values of P, $R_w \sim P^{-1}$ and $R_{ww} \sim P^{-1}$. This is a distinct improvement over the *a priori* bound of § 6, which was $\sim P^{-2}$. In fact it later develops that our numerical computation for R_E gives $R_E \sim P^{-1}$ as $P \to \infty$ as well. Thus, we have listed the values of PR_{ww} and PR_w as functions of P in table 1 as well as the values of R_{ww} and R_w. Figure 3 indicates R_w and R_{ww} as functions of P.

The method of generating lower bounds to R_E by relaxing the restriction $\nabla . \mathbf{u} = 0$ is not confined to the problem considered here, but could be used quite generally. The advantage is that one could quickly obtain the lower bound analytically merely by solving a Sturm–Liouville problem such as (7.5).

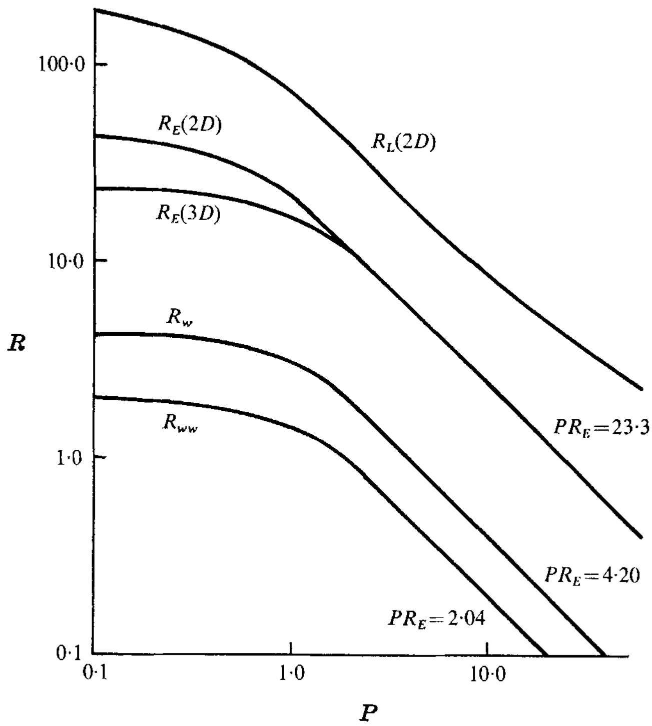

FIGURE 3. The stability boundaries and lower bounds for large P; $R_E(2D)$, $R_E(3D)$, R_w, R_{ww} and $R_L(2D)$.

8. Properties and solution of the Euler–Lagrange equations

8.1. *Reduction to ordinary differential equations*

Let us now express those functions in $\mathscr{F}$ as in (7.5). The Euler–Lagrange system (5.2), in these cases becomes the following:

$$\left.\begin{aligned} \tfrac{1}{2}RW_x\hat{w}+\tfrac{1}{2}RPT_x\hat{\phi} &= -D\hat{p}+(D^2-m^2)\,\hat{u},\\ 0 &= -ik\hat{p}+(D^2-m^2)\,\hat{v},\\ \tfrac{1}{2}RW_x\hat{u} &= -il\hat{p}+(D^2-m^2)\,\hat{w},\\ \tfrac{1}{2}RPT_x\hat{u} &= (D^2-m^2)\,\hat{\phi},\\ D\hat{u}+ik\hat{v}+il\hat{w} &= 0,\\ \hat{u}=\hat{v}=\hat{w}=\hat{\phi} &= 0 \quad \text{on} \quad x=0,\\ \hat{u},\hat{v},\hat{w},\hat{\phi} &\to 0 \quad \text{as} \quad x\to\infty, \end{aligned}\right\} \qquad (8.1\,a\text{–}g)$$

where $$D=\frac{d}{dx}.$$

For a given value of P and given values of k and l, let us denote the smallest positive eigenvalue of R of system (8.1) by $\tilde{R}(P; k, l)$.

8.2. *Symmetry properties*

For fixed P, $\tilde{R}(P; k, l)$ satisfies the following symmetry relations:

$$\left.\begin{aligned} \tilde{R}(P;\, k, l) &= \tilde{R}(P;\, -k, l), \\ \tilde{R}(P, k, l) &= \tilde{R}(P;\, k, -l). \end{aligned}\right\} \qquad (8.2a, b)$$

Relation (8.2*a*) can be obtained from the transformation,

$$(k, l, \hat{u}, \hat{v}, \hat{w}, \hat{\phi}, \hat{p}, \tilde{R}) \rightarrow (-k_1, l_1, u_1, -v_1, w_1, \phi_1, p_1, \bar{R})$$

of system (8.1). From this we see that $\tilde{R} = \bar{R}$. Similarly, relation (8.2*b*) can be obtained from the transformation

$$(k, l, \hat{u}, \hat{v}, \hat{w}, \hat{\phi}, \hat{p}, \tilde{R}) \rightarrow (k_1, -l_1, -u_1^*, v_1^*, -w_1^*, -\phi_1^*, -p_1^*, \bar{R})$$

of system (8.1), where the asterisk denotes the complex conjugate.

The symmetry conditions (8. 2) indicate that in the eventual numerical solution of system (8.1) it is sufficient that one search only the first quadrant of the $k-l$ plane in order to find the smallest positive eigenvalue of R.

8.3. *Squire's theorem for large Prandtl number limit*

In § 6 it was shown that for $P \rightarrow \infty$ that

$$PR_{ww} \rightarrow 2{\cdot}04, \quad R_E \geqslant R_{ww}.$$

In addition, Gill & Davey (1969) found that as $P \rightarrow \infty$,

$$R_L \rightarrow 0.$$

Since $R_L \geqslant R_E$ always, it follows that, as $P \rightarrow \infty$,

$$R_E \rightarrow 0.$$

Thus, the asymptotic form for large P of system (8.2) has the form,

$$\left.\begin{aligned} \tfrac{1}{2}C\, T_x \hat{\phi} &= -D\hat{p} + (D^2 - m^2)\,\hat{u}, \\ 0 &= -ik\hat{p} + (D^2 - m^2)\,\hat{v}, \\ 0 &= -il\hat{p} + (D^2 - m^2)\,\hat{w}, \\ \tfrac{1}{2}C\, T_x \hat{u} &= (D^2 - m^2)\,\hat{\phi}, \\ D\hat{u} + i(k\hat{v} + l\hat{w}) &= 0, \\ \hat{u} = \hat{v} = \hat{w} = \hat{\phi} &= 0 \text{ on } x = 0, \\ \hat{u}, \hat{v}, \hat{w}, \hat{\phi} &\rightarrow 0 \quad \text{as} \quad x \rightarrow \infty, \end{aligned}\right\} \qquad (8.3)$$

where the product $\tilde{R}P \rightarrow C$ (constant) as $P \rightarrow \infty$. For large P, then, the constant C becomes the eigenvalue. The asymptotic behaviour $\tilde{R} \sim CP^{-1}$ for large P is precisely the same as that found in § 6 and § 7 for the lower bounds R_w and R_{ww}.

A version of Squire's theorem can be found in the large Prandtl number limit. Consider the transformation of system (8.3):

$$\begin{aligned}\bar{k}\bar{v} &= k\hat{v}+l\hat{w},\\ \bar{k}^2 &= k^2+l^2,\\ (\bar{u},\bar{\phi},\bar{p}) &= (\hat{u},\hat{\phi},\hat{p}).\end{aligned}$$

The resulting system has the form

$$\begin{aligned}\tfrac{1}{2}CT_x\bar{\phi} &= -D\bar{p}+(D^2-m^2)\,\bar{u},\\ 0 &= -i\bar{k}\bar{p}+(D^2-m^2)\,\bar{v},\\ \tfrac{1}{2}CT_x\bar{u} &= (D^2-m^2)\bar{\phi},\\ D\bar{u}+i\bar{k}\bar{v} &= 0,\\ \bar{u}=\bar{v}=\bar{\phi} = 0 \quad &\text{on} \quad x=0,\\ \bar{u},\bar{v},\bar{w},\bar{\phi}\to 0 \quad &\text{as} \quad x\to 0.\end{aligned}$$

The above system gives the minimum positive eigenvalue of C as a function of $\bar{k} = (l^2+k^2)^{\frac{1}{2}}$. This system is precisely that of (8.3) with $l = 0$.

This result indicates that as $P\to\infty$ the $R_E(P)$ for two and the general three-dimensional problems should merge together.

8.4. *Numerical scheme*

Let us transform system (8.1) into a set of eight first-order, complex linear equations (appendix B) of the form,

$$Du_j = A_{jk}u_k+iB_{jk}u_k \quad (j=1,\ldots,8), \tag{8.4}$$

where we sum over repeated subscripts $k = 1,\ldots,8$, and A_{jk} and B_{jk} are real functions of x, k, l, R, P.

By decomposing u_j into real and imaginary parts,

$$u_j = S_j+iQ_j \quad (j=1,\ldots,8),$$

system (8.4) can now be written as a system of sixteen real, linear, first-order equations (appendix B),

$$\left.\begin{aligned}DS_j &= A_{jk}S_k-B_{jk}Q_k,\\ DQ_j &= A_{jk}Q_k+B_{jk}S_k.\end{aligned}\right\} \tag{8.5a,b}$$

Due to the coupling present in the decomposition of system (8.4) only half, or eight, need be found in order to determine the complete sixteen linearly independent solutions. To see this, let us denote $\{(S_j,Q_j)|\,j=1,2,\ldots,8\}$ by $\{(S_j,Q_j)\}$ and let $\{(\bar{S}_j,\bar{Q}_j)\}$ be a solution to (8.5). Then,

$$\{(\bar{\bar{S}}_j,\bar{\bar{Q}}_j)\} \equiv \{(\bar{Q}_j,-\bar{S}_j)\}$$

is also a solution. These can be seen to be linearly independent of the first set.

We wish to determine for fixed values of P, k, l the minimum positive value $\tilde{R}$ of R such that $\{(S_j,Q_j)\}$ satisfy (8.5) and the boundary conditions

$$\begin{aligned}S_j = Q_j = 0 \quad &\text{on} \quad x=0,\\ S_j,Q_j\to 0 \quad &\text{as} \quad x\to\infty \quad (j=1,\ldots,8).\end{aligned}$$

(We choose the first four S_j and Q_j to correspond to the real and imaginary parts of $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{\phi}$ respectively (see appendix B)).

If the correct eigenfunctions $\{(\tilde{S}_j, \tilde{Q}_j)\}$ were known, they would satisfy conditions at $x = 0$ of the form,

$$\left.\begin{aligned} &(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4, \tilde{S}_5, \tilde{S}_6, \tilde{S}_7, \tilde{S}_8) = (0, 0, 0, 0, \alpha_2, \alpha_4, \alpha_6, 0) \\ \text{and}\quad &(\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \tilde{Q}_4, \tilde{Q}_5, \tilde{Q}_6, \tilde{Q}_7, \tilde{Q}_8) = (0, 0, 0, 0, \alpha_1, \alpha_3, \alpha_5, 1). \end{aligned}\right\} \tag{8.6a, b}$$

This solution has been arbitrarily normalized to make $(\tilde{S}_j, \tilde{Q}_j) = (0, 1)$.

We can represent the $\{(\tilde{S}_j, \tilde{Q}_j)\}$ in terms of eight (in fact only four, as discussed above) independent solutions of initial value problems. The solutions to the initial value problems denoted by $\{(S_j^{(k)}, Q_j^{(k)}) | k = 1, 2, 3, 4\}$ satisfy the equations (8.5) and the following initial conditions:

$$\left.\begin{array}{lcccc} & k=1 & k=2 & k=3 & k=4 \\ S_j^{(k)}, j = 1, \ldots, 8 & 0 & 0 & 0 & 0 \\ Q_j^{(k)}, j = 1, \ldots, 4 & 0 & 0 & 0 & 0 \\ Q_5^{(k)} & 1 & 0 & 0 & 0 \\ Q_6^{(k)} & 0 & 1 & 0 & 0 \\ Q_7^{(k)} & 0 & 0 & 1 & 0 \\ Q_8^{(k)} & 0 & 0 & 0 & 1 \end{array}\right\} \tag{8.7}$$

In terms of these solutions, we can express the exact eigenfunctions as follows:

$$\left.\begin{aligned} \tilde{S}_j &= \alpha_1 S_j^{(1)} + \alpha_2 Q_j^{(1)} + \alpha_3 S_j^{(2)} + \alpha_4 Q_j^{(2)} + \alpha_5 S_j^{(3)} + \alpha_6 Q_j^{(3)} + S_j^{(4)} \\ \tilde{Q}_j &= \alpha_1 Q_j^{(1)} - \alpha_2 S_j^{(1)} + \alpha_3 Q_j^{(2)} - \alpha_4 S_j^{(2)} + \alpha_5 Q_j^{(3)} - \alpha_6 S_j^{(3)} + Q_j^{(4)} \end{aligned}\right\} \quad (j = 1, \ldots, 8). \tag{8.8}$$

We replace the boundary conditions for $x \to \infty$ by (appendix B)

$$\left.\begin{aligned} &M^3\tilde{S}_j = M^3\tilde{Q}_j = 0 \quad (j = 1, 2, 3) \quad \text{as} \quad x \to x_1, \\ \text{and}\quad &M\tilde{S}_4 = M\tilde{Q}_4 = 0, \qquad\qquad\quad \text{as} \quad x \to x_1. \end{aligned}\right\} \tag{8.9}$$

In reality the conditions should be applied as $x_1 \to \infty$. Numerically, it was found that $x_1 = 8$ was large enough that increasing its value left the value of $\tilde{R}$ unchanged in the fifth significant digit. The application of conditions (8.9) to (8.8) at $x = x_1$ yields eight inhomogeneous algebraic equations for the six unknowns α_i, $i = 1, \ldots, 6$ as follows:

$$\left.\begin{aligned} &\left.\begin{aligned} &\alpha_1 M^3S_j^{(1)} + \alpha_2 M^3Q_j^{(1)} + \alpha_3 M^3S_j^{(2)} + \alpha_4 M^3Q_j^{(2)} + \alpha_3 M^3S_j^{(3)} \\ &\qquad\qquad + \alpha_6 M^3Q_j^{(3)} = -M^3S_j^{(4)}, \\ &\alpha_1 MS_4^{(1)} + \alpha_2 MQ_4^{(1)} + \alpha_3 MS_4^{(2)} + \alpha_4 MQ_4^{(2)} + \alpha_5 MS_4^{(3)} \\ &\qquad\qquad + \alpha_6 MQ_4^{(3)} = -MS_4^{(4)}, \end{aligned}\right\} \quad (j = 1, \ldots, 3), \\ &\left.\begin{aligned} &\alpha_1 M^3Q_j^{(1)} - \alpha_2 M^3S_j^{(1)} + \alpha_3 M^3Q_j^{(2)} - \alpha_4 M^3S_j^{(2)} + \alpha_5 M^3Q_j^{(3)} \\ &\qquad\qquad - \alpha_6 M^3S_j^{(3)} = -M^3Q_j^{(4)}, \\ &\alpha_1 MQ_4^{(1)} - \alpha_2 MS_4^{(1)} + \alpha_3 MQ_4^{(2)} - \alpha_4 MS_4^{(2)} + \alpha_5 MQ_4^{(3)} \\ &\qquad\qquad - \alpha_6 MS_4^{(3)} = -MQ_4^{(4)}, \end{aligned}\right\} \quad (j = 1, \ldots, 3). \end{aligned}\right\} \tag{8.10}$$

For a given R, P, k and l the only unknowns are the α_i's.

We first solve for the α_i, using six of the equations ($j = 1, 2, 4$) and using another six ($j = 1, 3, 4$). In general, the two sets of α_i's are not compatible. They are only compatible if we have used the correct value of $R = \tilde{R}$.

Our numerical scheme is as follows:

(1) Fix P, k, l.

(2) Set $R = 0$.

(3) Numerically integrate equations (8.5) four times corresponding to initial conditions (8.7).

(4) Compute the coefficients of the α_i in (8.10) at some $x = x_1$.

(5) Calculate both sets (as defined above) of α_i. Denote the differences by $\delta\alpha_i$.

(6) Increase the value of R and repeat starting at (3).

(7) When R has been increased sufficiently, all the $\delta\alpha_i$ will change sign between successive guesses of R.

(8) Linear interpolation was used to find the correct value $\tilde{R}$ and also the correct α_i.

(9) With this correct $\tilde{R}$ and α_i, $i = 1, \ldots, 8$, equations (8.5) were integrated subject to initial conditions (8.6) in order to check that the correct asymptotic behaviour (8.9) was present.

(10) With the correct value of $\tilde{R}(P; k, l)$ computed (for fixed P) we sought $R_E = \inf\limits_{k,l} \tilde{R}(P; k, l)$ for $k, l \geqslant 0$.

The numerical integration used was the Runge–Kutta–Gill method (Ralston & Wilf 1960). The step size Δx was automatically adjusted by a subroutine to keep the integration difference between using Δx and $2\Delta x$ within a prescribed tolerance. This tolerance was chosen so that the value of $\tilde{R}$ showed no change in the third significant digit as the tolerance was decreased. The final value $x_1 = 8$ was used. No significant difference (fifth significant digit) in $\tilde{R}$ could be detected numerically by choosing a larger terminating value.

9. Results and conclusions

The object of the numerical analysis was to find the values $R_E(P)$. When $R < R_E(P)$, the buoyancy boundary layer is asymptotically stable in the mean over rectangular parallelepipeds bounded in x, and having the wall as one of its boundaries, and is additionally the unique steady solution of the governing equations over the class $\mathscr{F}$.

Easily accessible lower bounds to $R_E(P)$ were developed by solving the maximum problem (M) over a space of functions not restricted by $\nabla . \mathbf{u} = 0$. These lower bounds R_w turned out to be about a factor of six lower than R_E and can be obtained from the solution of a Sturm–Liouville problem. Simpler lower bounds R_{ww} were found analytically at about a factor of two below R_w. In both cases, the asymptotic behaviour at $P \rightarrow \infty$ of $R_w(P)$ and $R_{ww}(P)$ was P^{-1}, the same as that of $R_E(P)$. Lower bounds of this form are easily obtained in other applications of the energy theory but are especially useful here since a universal stability criterion (Serrin 1959) is not obtainable. This is due to the fact that the flow domain has no finite dimension.

The linear theory of Gill & Davey (1969) was restricted to two-dimensional disturbances ($k = 0$). The first numerical calculation for the energy limit R_E was for two-dimensional disturbances. These results are listed in table 2 and in figures 2 and 3 denoted by $R_E(2D)$. The asymptotic behaviour at $P \to \infty$ is $R_E(2D) \sim P^{-1}$, whereas Gill & Davey find that $R_L(2D) \sim P^{-\frac{1}{2}}$. For $0 \leqslant P \leqslant 10$, the values of $R_E(2D)$ are between one-third and one-fourth those of $R_L(2D)$.

For general three-dimensional disturbances it was sufficient due to the symmetry relations in § 8.2 to search the wave-number plane only in the first quadrant $k, l \geqslant 0$. It was found that R_E corresponded to $l = 0$ within numerical accuracy. All critical Reynolds numbers are accurate to within $\pm$ 1 in the third significant digit. Since for fixed P the curve $\tilde{R}(P; k, l)$ is very flat in the vicinity of R_E, the

	Energy, three-dimensional ($l = 0$)			Energy, two-dimensional ($k = 0$)			Linear, two-dimensional ($k = 0$)	
P	k	R_E	PR_E	l	R_E	PR_E	l	R_L
0·0	0·44	23·3	0·0	0·50	43·9	0·0	0·62	141
0·1	0·45	23·2	2·32	0·50	43·3	4·33	0·59	109
0·72	0·45	19·7	14·2	0·70	27·2	19·6	0·29	101
1·0	0·55	17·5	17·5	0·70	21·3	21·3	0·28	76·1
3·0	0·80	7·55	22·7	0·85	7·72	23·2	0·35*	25·5*
5·0	0·85	4·63	23·1	0·85	4·66	23·3	0·39	15·5*
10·0	0·85	2·33	23·3	0·85	2·33	23·3	0·44	8·50
∞	0·85	0·00	23·3	0·85	0·00	23·3	0·42	0·00

TABLE 2. Comparisons of $R_E(2D)$, $R_E(3D)$ and $R_L(2D)$ for various values of P. Asterisks indicate an interpolation of Gill & Davey's data.

accuracy of the critical wave-number is only to $\pm$ 0·05. However, it is plausible that R_E corresponds to $l = 0$ exactly since $l = 0$ is a surface of symmetry of $\tilde{R}$. As can be seen from table 2 and figures 2 and 3, for small P, $R_E(3D)$ differs from $R_E(2D)$ by a factor of two but for large P, say $P > 3$, the results of the two calculations merge. This was in fact predicted in § 8.3 where for $P \to \infty$, a Squire's theorem was valid. Thus, again $R_E(3D) \sim 23{\cdot}3P^{-1}$ for $P \to \infty$.

For a given Prandtl number P, subcritical instabilities of the buoyancy boundary layer are allowable when R satisfies $R_E < R < R_L$. It is clear in the buoyancy boundary layer for small values of P that R_E and R_L are of the same order of magnitude. It is in this region that the instability is of inflexional type so that the hope expressed in the introduction that the energy method could give physically interesting results in a shear flow instability is borne out. The region broadens greatly as P gets larger since R_E and R_L have different asymptotic behaviours. It is in this region that the instability is buoyancy driven. The computed region $R_E < R < R_L$ certainly contains all possible subcritical instabilities. However, the allowable region may in fact be smaller. First, the Gill & Davey result for R_L is only an upper bound to the linear theory critical value of R, since they consider only two-dimensional disturbances (no Squire's theorem holds). Secondly, the value of R_E computed herein may only be a lower bound to the optimal energy stability limit for R. This is possible since we confined our attention to the linking parameter λ equal to one.

The only experimental evidence is Elder's (1965) examination of the buoyancy boundary layer in a vertical slot. He obtained a crude empirical formula from his results of the form $R_c = 200P^{-\frac{1}{4}}$. This seems closer to the energy rather than the linear result asymptotically for large P but has too large a coefficient.

Although one should be cautious in attributing physical significance to the eigenfunctions of the energy theory (since they are solutions to Euler–Lagrange equations and not hydrodynamic equations), let us interpret these in physical terms. Since $R_E(3D)$ corresponds to $l = 0$, the energy theory selects a mode which has velocity and temperature fields independent of z, i.e. independent of the direction of the basic flow $W(x)$. The wavelength preferred is smaller than for linear theory for large P, while larger than linear theory for small P.

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Appendix A. Numerical scheme for the integration of the Euler–Lagrange equations without divergence-free restriction

The Euler–Lagrange equations (7.3) can be written as a system of six linear first-order differential equations after separation of variables (see above, equations (8.1)). The eigenfunctions $\hat{u}$, $\hat{w}$, $\hat{\phi}$ ($\hat{v} \equiv 0$) can be written as a linear combination of three linearly independent solutions

$$(\hat{u}, \hat{w}, \hat{\phi}) = a(\hat{u}_1, \hat{w}_1, \hat{\phi}_1) + b(\hat{u}_2, \hat{w}_2, \hat{\phi}_2) + c(\hat{u}_3, \hat{w}_3, \hat{\phi}_3), \tag{A 1}$$

where the independent functions satisfy the following boundary conditions at $x = 0$:

$$\left.\begin{array}{lll} \hat{u}_1 = 0, & \hat{u}_2 = 0, & \hat{u}_3 = 0, \\ \hat{w}_1 = 0, & \hat{w}_2 = 0, & \hat{w}_3 = 0, \\ \hat{\phi}_1 = 0, & \hat{\phi}_2 = 0, & \hat{\phi}_3 = 0, \\ D\hat{u}_1 = 1, & D\hat{u}_2 = 0, & D\hat{u}_3 = 0, \\ D\hat{w}_1 = 0, & D\hat{w}_2 = 1, & D\hat{w}_3 = 0, \\ D\hat{\phi}_1 = 0, & D\hat{\phi}_2 = 0, & D\hat{\phi}_3 = 1. \end{array}\right\} \tag{A 2}$$

It can be seen from (A 1) and (A 2) that

$$\hat{u}, \hat{w}, \hat{\phi} = 0 \quad \text{on} \quad x = 0.$$

Asymptotically, for large x, we want $\hat{u}$, $\hat{w}$, $\hat{\phi}$ to approach zero and it can be seen from (7.3) that the decaying solution is given by

$$\hat{u}, \hat{w}, \hat{\phi} \sim e^{-mx}.$$

We therefore define M as follows:

$$M = D + m$$

and we take as the boundary condition for large x that

$$M\hat{u}, \quad M\hat{w}, \quad M\hat{\phi} \to 0 \quad \text{as} \quad x \to \infty. \tag{A 3}$$

If we apply conditions (A 3) to (A 1) and require a non-trivial solution, we obtain the condition that

$$\Delta = \begin{vmatrix} M\hat{u}_1 & M\hat{u}_2 & M\hat{u}_3 \\ M\hat{w}_1 & M\hat{w}_2 & M\hat{w}_3 \\ M\hat{\phi}_1 & M\hat{\phi}_2 & M\hat{\phi}_3 \end{vmatrix} = 0 \quad \text{at} \quad x = x_1.$$

The eigenvalue problem reduces to finding the smallest positive value R_w of R for which $\Delta \to 0$ as $x_1 \to \infty$. The procedure for a fixed $m > 0$, is to start with $R = 0$ and to increase it until Δ changes sign (for x_1 fixed). The value of $R_w^{(m)}$ is found by repeated linear interpolation. The procedure was repeated for a different positive m and the $\inf\limits_m R_w^{(m)}$ was calculated. As in the case of computing R_{ww}, the $\inf\limits_m R_w^{(m)}$ was approached asymptotically as $m \to 0$. Various values of the terminating value x_1 were tried and it was found that the values R_w and R_{ww} were insensitive for the value of x_1 for $x_1 \geqslant 8$.

Appendix B. System of real, linear, first-order equations employed in solution of the eigenvalue problem

If we define the variables

$$\left.\begin{aligned} \hat{a} &= (D-m)\,\hat{v}, \\ \hat{b} &= (D-m)\,\hat{w}, \\ \hat{c} &= (D-m)\,\hat{\phi}, \end{aligned}\right\} \tag{B 1a–c}$$

the Euler–Lagrange equations (8.1a–e) can be written as a system of eight complex linear equations as follows:

$$\left.\begin{aligned} D\hat{u} &= -ik\hat{v} - il\hat{w}, \\ D\hat{v} &= \hat{a} + m\hat{v}, \\ D\hat{w} &= \hat{b} + m\hat{w}, \\ D\hat{\phi} &= \hat{c} + m\hat{\phi}, \\ D\hat{a} &= ik\hat{p} - m\hat{a}, \\ D\hat{b} &= il\hat{p} - m\hat{b} + \tfrac{1}{2}RW_x\hat{u}, \\ D\hat{c} &= -m\hat{c} + \tfrac{1}{2}PRT_x\hat{u}, \\ D\hat{p} &= -\tfrac{1}{2}RW_x\hat{w} - \tfrac{1}{2}PRT_x\hat{\phi} - ik\hat{a} - il\hat{b} - ikm\hat{v} - ilm\hat{w} - m^2\hat{u}. \end{aligned}\right\} \tag{B 2}$$

This complex system can be converted to a real system of sixteen equations by breaking into real and imaginary parts.

Appendix C. Asymptotic form of the eigenfunctions as $x \to \infty$

For $x \to \infty$, the terms of the Euler–Lagrange system (8.1) containing W_x and T_x approach zero exponentially so that the asymptotic equations are given as follows:

$$\left.\begin{aligned} 0 &= -D\hat{p} + (D^2 - m^2)\,\hat{u}, \\ 0 &= -ik\hat{p} + (D^2 - m^2)\,\hat{v}, \\ 0 &= -il\hat{p} + (D^2 - m^2)\,\hat{w}, \\ 0 &= (D^2 - m^2)\,\hat{\phi}, \\ 0 &= D\hat{u} + ik\hat{v} + il\hat{w}, \\ \hat{u}, \hat{v}, \hat{w}, \hat{\phi} &\to 0 \text{ as } x \to \infty. \end{aligned}\right\} \qquad (\mathrm{C}\ 1\,a\text{–}f)$$

In this limit, the temperature disturbance $\hat{\phi}$ become decoupled from the velocity disturbance, and we have

$$\hat{\phi} \sim e^{\pm mx}.$$

The decaying solution thus satisfies

$$M\hat{\phi} \to 0 \quad \text{as} \quad x \to \infty,$$

where

$$M \equiv D + m. \qquad (\mathrm{C}\ 2)$$

The remaining variables are governed by a linear, complex system of constant coefficient equations whose solutions have the form,

$$(\hat{u}, \hat{v}, \hat{w}, \hat{p}) = e^{\hat{\nu}x} \quad (u_0, v_0, w_0, p_0),$$

where the subscripted quantities are constants. The substitution of this into (C 1) yields a non-trivial solution if the determinant of coefficients vanishes. The result of this is as follows:

$$(\nu^2 - m^2)^3 = 0.$$

For large x, the real and imaginary parts of $(\hat{u}, \hat{v}, \hat{w}, \hat{p})$ must satisfy

$$M^3(\hat{u}, \hat{v}, \hat{w}, \hat{p}) \to 0, \qquad (\mathrm{C}\ 3)$$

in order that (C $1f$) hold.

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